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A geometrically nonlinear analysis of coplanar crack propagation in some heterogeneous medium

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ABSTRACT

In a recent paper (Leblond et al., 2012), we established, using some results of Rice (1989), the second-order expression of the variation of the mode I stress intensity factor resulting from some small, but otherwise arbitrary coplanar perturbation of the front of a semi-infinite tensile crack in an infinite body. The aim of the present work is to apply the expression found to a geometrically nonlinear analysis of quasistatic, coplanar crack propagation in some heterogeneous medium. In a first step, we recall Leblond et al. (2012)'s formula, extending it to the case where the unperturbed stress intensity factor, for the straight configuration of the front, depends on the position of this front; in addition to being intrinsically interesting, such an extension is necessary in order to avoid meaningless divergent integrals in what follows. In a second step, assuming the local energy-release-rate to be equal everywhere on the crack front to its critical value, we derive an expression of the shape of this front accurate to second order in the fluctuations of toughness of the material. In a third step, as an application, we present a second-order calculation of the equilibrium shape of the crack front, when it penetrates a single infinitely elongated obstacle or a periodic distribution of such obstacles. Special attention is paid to the case, of particular physical interest, where the derivative of the unperturbed stress intensity factor with respect to the position of the crack front can be neglected.

1. Introduction

A quarter of a century ago, Rice (1985) derived the first-order expression of the variation of the mode I stress intensity factor (SIF) induced by some small, but otherwise arbitrary coplanar perturbation of the front of a semi-infinite tensile crack in an infinite body. This expression has been extensively used since to investigate the behavior of cracks propagating in heterogeneous materials; see the works of Schmittbuhl et al. (1995), Tanguy et al. (1998), Schmittbuhl and Vilotte (1999), Krishnamurthy et al. (2000), Roux et al. (2003), Schmittbuhl et al. (2003), Charles et al. (2004), Katzav and Adda-Bedia (2006), Bonamy et al. (2008), Ponson (2009), Laurson et al. (2010), Ponson and Bonamy (2010), among others, and the reviews of Alava et al. (2006) and Lazarus (2011). Quite recently, Legrand et al. (2011) extended Rice (1985)'s formula to the case of coplanar perturbation of an emerging tensile crack lying on the mid-plane of a plate, thus accounting for the effect of the finite dimensions of the specimens; and Patinet et al. (2011) showed that use of the new formula did lead to improved agreement of experimental and computed shapes of crack fronts deformed by the presence of hard obstacles.

However Rice (1985)'s and Legrand et al. (2011)'s formulae were accurate only to first order in the perturbation of the front, whereas geometric nonlinearities could be suspected to play a significant role in actual experimental situations. This was the motivation for Leblond et al. (2012)'s very recent extension to second order of Rice (1985)'s first-order formula for coplanar perturbation of a semi-infinite crack in an infinite body. Leblond et al. (2012)'s treatment was basically simple and relied on application of some general results of Rice (1989). Their formula for the perturbed SIF was found to differ from the earlier ones, themselves in conflict, of Adda-Bedia et al. (2006) and Katzav et al. (2007), derived by a more complex method. Direct numerical computations of the SIF along perturbed crack fronts by the finite element method confirmed the correctness of Leblond et al. (2012)'s new formula.

The aim of this paper is to apply Leblond et al. (2012)'s formula to some geometrically nonlinear analysis of quasistatic, coplanar crack propagation in media having a heterogeneous distribution of fracture toughness. We shall first extend this formula to the case where the unperturbed SIF, for the straight configuration of the front, depends on the position of this front within the crack plane. Such an extension is desirable to address realistic geometrical conditions. In addition, even the interesting case where the unperturbed SIF becomes independent of the position of the front requires this extended formula. The reason is that considering

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immediately this case would generate meaningless divergent integrals in both the first- and second-order expressions of the equilibrium shape of the crack front in heterogeneous materials. However, once expressions involving the derivatives of the unperturbed SIF with respect to the position of the front are obtained, it becomes possible, with certain precautions, to consider the limiting case where these derivatives go to zero.

The paper is organized as follows:

- Section 2 recalls Leblond et al. (2012)'s results, suitably extended as just explained. The presentation is brief since the extension is simple and again based on straightforward use of Rice (1989)'s results.
- Section 3 applies the results found to the case of a crack propagating quasistatically, according to Griffith's criterion, along a plane having a given heterogeneous distribution of fracture toughness. Assuming the energy-release-rate (ERR) to be equal to its critical value at every point of the crack front, we determine the resulting shape of this front up to second order in the fluctuations of toughness.
- As an application, Section 4 considers the case of a crack penetrating into a single obstacle of infinite length in the direction of propagation, or some periodic distribution of such obstacles. The equilibrium shape of the front is calculated up to second order in the contrast of toughness between the matrix and the obstacle (s). The first-order expression agrees with that found by Chopin (2010) while the second-order one is new.

2. Second-order coplanar perturbation of a semi-infinite crack

2.1. Notations

Consider (Fig. 1) a semi-infinite tensile crack located in some infinite body subjected to prescribed forces only (no prescribed displacements). Assume that the crack front is slightly curved, its equation in the crack plane Oxz being of the form

$$x(z) = a + \epsilon\phi(z) \tag{1}$$

where a denotes the distance from the axis Oz to some "reference straight front", ϵ a small parameter, and $\phi(z)$ a given, fixed, smooth function. The position of the front is thus characterized by the parameters a and ϵ, and the position of a current point along it by the parameter z.

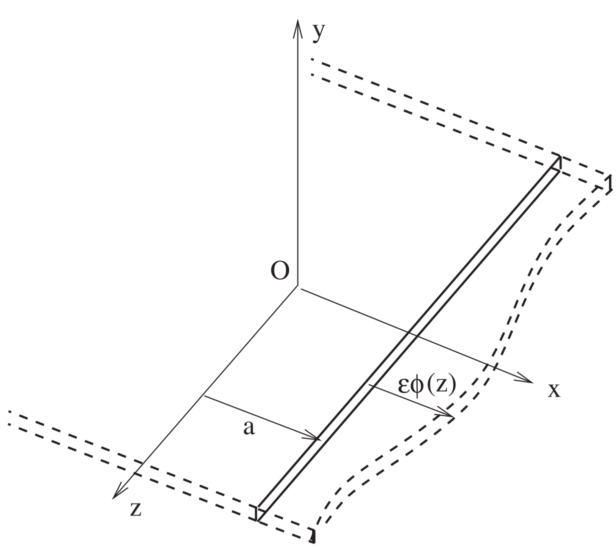

Fig. 1. A semi-infinite crack with a slightly perturbed front in an infinite body.

The mode I SIF for a given, fixed loading imposed upon this cracked geometry is denoted $K(a,\epsilon;z_1)$. Our interest lies in the second-order expression of this SIF with respect to ϵ:

$$K(a,\epsilon;z) \equiv K^0(a) + \epsilon K^1(a;z) + \epsilon^2 K^2(a;z) + O(\epsilon^3). \tag{2}$$

It is assumed in this equation that the loading has a translatory invariance in the direction z, so that the unperturbed SIF $K^0(a)$ depends on the position a of the (straight) front but not on the position of the point of observation along it.

2.2. Second-order expansion of the stress intensity factor in the physical space

At order 1, $K^1(a;z)$ is obtained through direct application of Rice (1989)'s formula for the variation of the SIF to the straight configuration of the front; this is possible since the *fundamental kernel* appearing in this formula is known explicitly for this configuration. The result reads:

$$K^1(a;z_1) = \frac{dK^0}{da}(a)\phi(z_1) + \frac{K^0(a)}{2\pi} PV \int_{-\infty}^{-\infty} \frac{\phi'(z)}{z - z_1} dz \tag{3}$$

where the symbol PV denotes a Cauchy principal value.

At order 2, $K^2(a;z)$ may again be obtained from the same formula of Rice (1989), now applying it to some pre-perturbed configuration of the front upon which a secondary, *infinitesimal* and *proportional* perturbation is superimposed. When doing so, one must use formulae for the SIF and the fundamental kernel on the pre-perturbed configuration, accurate to first order in the primary perturbation; the first of these formulae is provided by Eq. (3) and the second by another equation of (Rice, 1989). The output is an expression of the derivative of the SIF with respect to the amplitude of the perturbation, accurate to first order in this amplitude; the second-order expression of the SIF then follows through integration. This was the procedure followed by Leblond et al. (2012); they assumed $K^0(a)$ to be independent of a, but this hypothesis is easily removed and the final result reads:

$$\begin{aligned} K^2(a;z_1) = {} & \frac{1}{2}\frac{d^2K^0}{da^2}(a)[\phi(z_1)]^2 + \frac{1}{2\pi}\frac{dK^0}{da}(a) PV \int_{-\infty}^{-\infty} \frac{\phi(z)\phi'(z)}{z - z_1} dz \\ & + \frac{K^0(a)}{8\pi^2} PV \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[\left(\frac{1}{z' - z_1} + \frac{2}{z' - z} \right)\phi'(z') \right. \\ & \left. + \frac{2}{z - z_1}\left(\frac{1}{z' - z_1} - \frac{1}{z' - z} \right)\phi(z') \right] \frac{\phi(z) - \phi(z_1)}{(z - z_1)^2} dz dz'. \end{aligned} \tag{4}$$

It is worth noting that for the sinusoidal perturbation

$$\phi(z) \equiv \cos(kz) \quad (k > 0),$$

Eq. (4) yields, after some calculations,

$$K^2(a;z) = \frac{1}{2}\frac{d^2K^0}{da^2}(a)\cos^2(kz) - \frac{1}{4}\frac{dK^0}{da}(a)k\cos(2kz) - \frac{K^0(a)}{8}k^2\sin^2(kz);$$

this result was also very recently arrived at by Willis (2012), by a completely different method.

2.3. Second-order expansions of the stress intensity factor and the energy-release-rate in Fourier's space

Expressions of the Fourier transforms of the SIF and the ERR in the direction of the crack front will be needed. The definition adopted here for the Fourier transform $\widehat{\psi}(k)$ of an arbitrary function $\psi(z)$ is

$$\psi(z) \equiv \int_{-\infty}^{+\infty} \widehat{\psi}(k) e^{ikz} dk \Longleftrightarrow \widehat{\psi}(k) \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} \psi(z) e^{-ikz} dz. \tag{5}$$

The expression of the Fourier transform $\widehat{K^1}(a;k)$ of $K^1(a;z)$ is then (Lazarus, 2011):

$$\hat{K^1}(a;k) = K^0(a)\left[\frac{dK^0}{K^0 da}(a) - \frac{|k|}{2}\right]\hat{\phi}(k). \quad (6)$$

That of the Fourier transform $\hat{K^2}(a;k)$ of $K^2(a;z)$ was derived by Leblond et al. (2012), again assuming $K^0(a)$ to be independent of a; but again this hypothesis is easily removed and the result reads:

$$\hat{K^2}(a;k_1) = K^0(a)\int_{-\infty}^{+\infty} P(a;k,k_1-k)\hat{\phi}(k)\hat{\phi}(k_1-k)dk \quad (7)$$

where

$$P(a;k,k') \equiv \frac{1}{2}\frac{d^2K^0}{K^0da^2}(a) - \frac{1}{4}\frac{dK^0}{K^0da}(a)|k+k'| + \frac{1}{16}\left\{\mathrm{sgn}(kk')(k+k')^2 + [\mathrm{sgn}(k)-\mathrm{sgn}(k')]|k+k'|(k-k') - k^2 - k'^2\right\}. \quad (8)$$

The expansion of the Fourier transform $\hat{G}(a,\epsilon;k)$ of the ERR $G(a,\epsilon;z)$ follows from Irwin's formula and the expressions (6) and (7) of $\hat{K^1}(a;k)$ and $\hat{K^2}(a;k)$:

$$\hat{G}(a,\epsilon;k) \equiv G^0(a)\delta(k) + \epsilon\hat{G^1}(a;k) + \epsilon^2\hat{G^2}(a;k) + O(\epsilon^3) \quad (9)$$

where δ denotes Dirac's function, $G^0(a)$ the unperturbed ERR, and

$$\begin{cases} \hat{G^1}(a;k) = G^0(a)\left[\frac{dG^0}{G^0da}(a) - |k|\right]\hat{\phi}(k) \\ \hat{G^2}(a;k_1) = G^0(a)\int_{-\infty}^{+\infty} Q(a;k,k_1-k)\hat{\phi}(k)\hat{\phi}(k_1-k)dk \end{cases} \quad (10)$$

with

$$Q(a;k,k') \equiv \frac{1}{2}\frac{d^2G^0}{G^0da^2}(a) - \frac{1}{4}\frac{dG^0}{G^0da}(a)(|k+k'|+|k|+|k'|) + \frac{1}{8}\left\{\mathrm{sgn}(kk')(k+k')^2 + [\mathrm{sgn}(k)-\mathrm{sgn}(k')]|k+k'|(k-k') - (|k|-|k'|)^2\right\}. \quad (11)$$

3. The equilibrium shape of the front of a crack propagating in a heterogeneous material

3.1. Generalities

We shall now apply the preceding results to the study of coplanar propagation of the crack governed by Griffith's criterion with a heterogenous fracture toughness $G_c(x,z)$ given by

$$G_c(x,z) \equiv \overline{G_c}[1+\epsilon g_c(x,z)], \quad (12)$$

where $\overline{G_c}$ is a "mean toughness", ϵ a small parameter and $g_c(x,z)$ a given function describing the toughness fluctuations. For a given loading, provided that G is equal to G_c at every point of the crack front, the distribution of toughness determines the shape of this front in the form

$$x = a + \epsilon\phi^1(a;z) + \epsilon^2\phi^2(a;z) + O(\epsilon^3) \quad (13)$$

where $a, \phi^1(a;z)$ and $\phi^2(a;z)$ are a parameter and functions to be determined.

3.2. First- and second-order expressions of the crack front shape

To determine $a, \phi^1(a;z)$ and $\phi^2(a;z)$, it is useful to rewrite, for a crack front shape of type (1), Eq. (10) in the form

$$\hat{G^1}(a;k) \equiv \hat{\mathcal{G}}^1\left[a;\{\hat{\phi}\}\right](k); \quad \hat{G^2}(a;k) \equiv \hat{\mathcal{G}}^2\left[a;\{\hat{\phi}\}\right](k). \quad (14)$$

These somewhat formal equalities express the fact that the functions $\hat{G^1}(a;\cdot)$ and $\hat{G^2}(a;\cdot)$ are linear and quadratic functionals, respectively, of the Fourier transform $\hat{\phi}$ of the "perturbation function" ϕ. With these notations, for the crack front shape depicted by Eq. (13), corresponding to the perturbation function $\phi \equiv \phi^1 + \epsilon\phi^2$, the expansion (9) of $\hat{G}(a,\epsilon;k)$ takes the form

$$\begin{aligned}\hat{G}(a,\epsilon;k) &= G^0(a)\delta(k) + \epsilon\hat{\mathcal{G}}^1\left[a;\{\hat{\phi^1}+\epsilon\hat{\phi^2}\}\right](k) + \epsilon^2\hat{\mathcal{G}}^2\left[a;\{\hat{\phi^1}+\epsilon\hat{\phi^2}\}\right](k) + O(\epsilon^3) \\ &= G^0(a)\delta(k) + \epsilon\hat{\mathcal{G}}^1\left[a;\{\hat{\phi^1}\}\right](k) + \epsilon^2\left\{\hat{\mathcal{G}}^1\left[a;\{\hat{\phi^2}\}\right](k) + \hat{\mathcal{G}}^2\left[a;\{\hat{\phi^1}\}\right](k)\right\} + O(\epsilon^3). \end{aligned} \quad (15)$$

Now the local toughness is

$$\begin{aligned} &G_c[x = a + \epsilon\phi^1(a;z) + \epsilon^2\phi^2(a;z) + O(\epsilon^3), z] \\ &= \overline{G_c}\{1 + \epsilon g_c[a + \epsilon\phi^1(a;z) + \epsilon^2\phi^2(a;z) + O(\epsilon^3), z]\} \\ &= \overline{G_c}\left[1 + \epsilon g_c(a,z) + \epsilon^2\frac{\partial g_c}{\partial x}(a,z)\phi^1(a;z)\right] + O(\epsilon^3). \end{aligned}$$

The spatial expansion of G_c here is based on the assumption that the geometrical perturbations of the crack front are small compared to the typical scale over which the local toughness varies in the direction of propagation. The Fourier transform of this expression at the point k_1 is

$$\overline{G_c}\left[\delta(k_1) + \epsilon\hat{g_c}(a,k_1) + \epsilon^2\int_{-\infty}^{+\infty}\frac{\partial\hat{g_c}}{\partial x}(a,k)\hat{\phi^1}(a;k_1-k)dk\right] + O(\epsilon^3).$$

Assuming G to be equal to G_c at every point of the crack front and therefore equating the right-hand side of Eq. $(15)_2$ (at $k = k_1$) to this expression, one gets the following conditions:

- At order 0:

$$G^0(a) = \overline{G_c}. \quad (16)$$

This condition determines the position a of the reference straight front as a function of the loading applied.

- At order 1:

$$\hat{\mathcal{G}}^1\left[a;\{\hat{\phi^1}\}\right](k) = \overline{G_c}\hat{g_c}(a,k),$$

which implies, by the expression $(10)_1$ of the functional $\hat{\mathcal{G}}^1\left[a;\{\hat{\phi^1}\}\right](k)$ and Eq. (16), that

$$\hat{\phi^1}(a;k) = -\frac{\hat{g_c}(a,k)}{|k| - \frac{dG^0}{G^0da}(a)}. \quad (17)$$

- At order 2:

$$\hat{\mathcal{G}}^1\left[a;\{\hat{\phi^2}\}\right](k_1) = -\hat{\mathcal{G}}^2\left[a;\{\hat{\phi^1}\}\right](k_1) + \overline{G_c}\int_{-\infty}^{+\infty}\frac{\partial\hat{g_c}}{\partial x}(a,k) \times \hat{\phi^1}(a;k_1-k)dk,$$

which implies, by Eqs. (10), (16) and (17), that

$$\widehat{\phi^2}(a;k_1) = \frac{1}{|k_1| - \frac{dG^0}{G^0da}(a)}\left\{\int_{-\infty}^{+\infty} Q(a;k,k_1-k)\frac{\hat{g_c}(a,k)}{|k| - \frac{dG^0}{G^0da}(a)}\frac{\hat{g_c}(a,k_1-k)}{|k_1-k| - \frac{dG^0}{G^0da}(a)}dk + \int_{-\infty}^{+\infty}\frac{\partial\hat{g_c}}{\partial x}(a,k)\frac{\hat{g_c}(a,k_1-k)}{|k_1-k| - \frac{dG^0}{G^0da}(a)}dk\right\}. \quad (18)$$

3.3. Comments

It is necessary to discuss here the convergence of the integrals appearing in the expressions of $\hat{\phi}^1(a;k)$ and $\hat{\phi}^2(a;k)$ and their inverse Fourier transforms $\phi^1(a;z)$ and $\phi^2(a;z)$. The following hypothesis is introduced on the sign of the derivative of the unperturbed ERR $G^0(a)$:

$$\frac{dG^0}{da}(a) < 0. \tag{19}$$

This condition is necessary for quasistatic propagation to be stable, as was the case in all experiments mentioned in the Introduction.

With this condition, the denominator in the expression (17) of $\hat{\phi}^1(a;k)$ can never be zero, so that the integral over k expressing $\phi^1(a;z)$ in terms of $\hat{\phi}^1(a;k)$ is perfectly convergent. Also, in the integrals appearing in the expression (18) of $\hat{\phi}^2(a;k_1)$, the denominators $|k| - \frac{dG^0}{G^0da}(a)$ and $|k_1 - k| - \frac{dG^0}{G^0da}(a)$ again cannot vanish so they do not raise any problem of convergence either. Finally the integral over k_1 expressing $\phi^2(a;z)$ in terms of $\hat{\phi}^2(a;k_1)$ involves the denominator $|k_1| - \frac{dG^0}{G^0da}(a)$, but again this denominator can never be zero and convergence of the integral is ensured.

Consider now the special but interesting situation where G^0 becomes independent of a. This is in fact a limit-case in which the derivative $\frac{dG^0}{da}(a)$ takes very small negative values; in physical terms, this means that the characteristic distance of variation of the unperturbed ERR is much larger than that of fluctuations of the fracture toughness.

When $\frac{dG^0}{da}(a) = 0$, the expression (17) of $\hat{\phi}^1(a;k)$ still makes sense, but the integral over k expressing $\phi^1(a;z)$ in terms of $\hat{\phi}^1(a;k)$ involves the denominator $|k|$ which becomes zero in the interval of integration; hence the integral diverges, except if $\hat{g_c}(a,0) = 0$. Also, in the expression (18) of $\hat{\phi}^2(a;k_1)$, the integrals are also divergent, except again if $\hat{g_c}(a,0) = 0$, because they involve the denominators $|k|$ and $|k_1 - k|$. Finally, even if $\hat{g_c}(a,0) = 0$, the expression of $\hat{\phi}^2(a;k_1)$ makes sense but that of $\phi^2(a;z)$ is once more a divergent integral over k_1 involving the denominator $|k_1|$.

It may thus be concluded that consideration of the limiting case where $\frac{dG^0}{da}(a) = 0$ requires some mathematical precautions. The best way to handle it is to first assume $\frac{dG^0}{da}(a) < 0$, perform all calculations with this hypothesis, and only finally examine whether the expressions found make sense in the limit $\frac{dG^0}{da}(a) \to 0$, which they may or may not, depending on the quantity of interest.

4. The shape of crack fronts encountering obstacles

As an application, we shall determine the equilibrium shape of the front of a crack penetrating into a single obstacle of infinite length in the direction of propagation, or a periodic distribution of such obstacles, up to second order in the contrast of toughness. The toughness of the matrix will be denoted $G_c^{\rm M}$, and the toughness and width of the obstacle (s), $G_c^{\rm O}$ and $2d$, respectively. We shall be particularly interested in the limit-case where $\frac{dG^0}{da}(a) \to 0$; this corresponds, in physical terms, to the situation where the typical distance of variation of the unperturbed ERR is much larger than d (and the period in the periodic case).

4.1. Case of a single obstacle

The distribution of toughness in this case is represented in Fig. 2. This distribution may be represented by formula (12) with

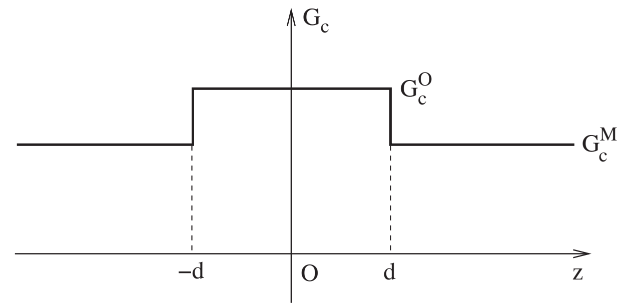

Fig. 2. The distribution of fracture toughness in an infinite body containing a single infinitely elongated obstacle.

$$\overline{G_c} \equiv G_c^{\rm M}; \quad \epsilon \equiv \frac{G_c^{\rm O} - G_c^{\rm M}}{G_c^{\rm M}}; \quad g_c(x,z) \equiv g_c(z) \equiv \begin{cases} 1 & \text{if } |z| < d \\ 0 & \text{if } |z| > d. \end{cases} \tag{20}$$

The Fourier transform of the function $g_c(x,z)$ is given by

$$\hat{g_c}(x,k) \equiv \hat{g_c}(k) \equiv \frac{1}{2\pi}\int_{-d}^{d} e^{-ikz}dz = \frac{\sin(kd)}{\pi k}. \tag{21}$$

At order 1, one gets from Eqs. (17) and (21):

$$\hat{\phi}^1(a;k) = -\frac{\sin(kd)}{\pi k\left[|k| - \frac{dG^0}{G^0da}(a)\right]}$$

so that

$$\begin{aligned}\phi^1(a;z) &= -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(kd)}{k\left[|k| - \frac{dG^0}{G^0da}(a)\right]}e^{ikz}dk \\ &= -\frac{2}{\pi}\int_0^{+\infty}\frac{\sin(kd)}{k\left[k - \frac{dG^0}{G^0da}(a)\right]}\cos(kz)dk.\end{aligned}$$

Now let $\frac{dG^0}{da}(a) \to 0$. It is clear that the integral defining $\phi^1(a;z)$ diverges in this limit; this is an illustration of the mathematical difficulties mentioned in Section 3.3. However we are not really interested in the absolute position of the crack front in the direction x, but only in its deviation from straightness. This deviation is characterized at order 1 by the quantity

$$\begin{aligned}\widetilde{\phi^1}(a;z) &\equiv \phi^1(a;z) - \phi^1(a;0) \\ &= \frac{2}{\pi}\int_0^{+\infty}\frac{\sin(kd)}{k\left[k - \frac{dG^0}{G^0da}(a)\right]}[1-\cos(kz)]dk.\end{aligned} \tag{22}$$

This quantity has a well-defined limit $\widetilde{\phi^1}(z)$ for $\frac{dG^0}{da}(a) \to 0$ given by

$$\widetilde{\phi^1}(z) \equiv \frac{2}{\pi}\int_0^{+\infty}\frac{\sin(kd)}{k^2}[1-\cos(kz)]dk. \tag{23}$$

To calculate this integral explicitly, it suffices, following Chopin (2010), to differentiate it with respect to z, evaluate the derivative using Gradshteyn and Ryzhik (1980)'s formula 3.741.1 and re-integrate. The result is (Chopin, 2010):

$$\widetilde{\phi^1}(z) = \frac{d}{\pi}[(1+u)\ln(|1+u|) + (1-u)\ln(|1-u|)], \quad u \equiv \frac{z}{d}. \tag{24}$$

At order 2, Eqs. (18) and (21) yield, since the function $\hat{g_c}(x,k)$ is independent of x:

$$\phi^2(a;z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q(a;k,k')\frac{\hat{g}_c(k)}{|k|-\frac{dG^0}{G^0da}(a)}\frac{\hat{g}_c(k')}{|k'|-\frac{dG^0}{G^0da}(a)}\frac{e^{i(k+k')z}}{|k+k'|-\frac{dG^0}{G^0da}(a)}dkdk'$$
$$= \frac{1}{\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q(a;k,k')\frac{\sin(kd)}{k\left[|k|-\frac{dG^0}{G^0da}(a)\right]}\frac{\sin(k'd)}{k'\left[|k'|-\frac{dG^0}{G^0da}(a)\right]}$$
$$\times\frac{e^{i(k+k')z}}{|k+k'|-\frac{dG^0}{G^0da}(a)}dkdk' = \frac{2}{\pi^2}\int\int_{k+k'\geqslant 0} Q(a;k,k')\frac{\sin(kd)}{k\left[|k|-\frac{dG^0}{G^0da}(a)\right]}\frac{\sin(k'd)}{k'\left[|k'|-\frac{dG^0}{G^0da}(a)\right]}$$
$$\times\frac{\cos[(k+k')z]}{k+k'-\frac{dG^0}{G^0da}(a)}dkdk'$$

where we have grouped the terms (k,k') and $(-k,-k')$ in the double integral and accounted for the fact that $Q(a;-k,-k') = Q(a;k,k')$, see Eq. (11).

Again, we are interested only in the deviation of the crack front from straightness, characterized at order 2 by the quantity

$$\widetilde{\phi^2}(a;z) \equiv \phi^2(a;z) - \phi^2(a;0) = \frac{2}{\pi^2}\int\int_{k+k'\geqslant 0} Q(a;k,k')\frac{\sin(kd)}{k\left[|k|-\frac{dG^0}{G^0da}(a)\right]}$$
$$\times\frac{\sin(k'd)}{k'\left[|k'|-\frac{dG^0}{G^0da}(a)\right]}\frac{\cos[(k+k')z]-1}{k+k'-\frac{dG^0}{G^0da}(a)}dkdk'. \quad (25)$$

This quantity has a well-defined limit $\widetilde{\phi^2}(z)$ for $\frac{dG^0}{da}(a)\to 0$ given by

$$\widetilde{\phi^2}(z) \equiv \frac{2}{\pi^2}\int\int_{k+k'\geqslant 0} Q^0(k,k')\frac{\sin(kd)}{k|k|}\frac{\sin(k'd)}{k'|k'|}$$
$$\times\frac{\cos[(k+k')z]-1}{k+k'}dkdk' \quad (26)$$

where

$$Q^0(k,k') \equiv \lim_{dG^0/da\to 0} Q(a;k,k') = \frac{1}{8}\left\{\mathrm{sgn}(kk')(k+k')^2\right.$$
$$\left.+\left[\mathrm{sgn}(k)-\mathrm{sgn}(k')\right]|k+k'|(k-k') - \left(|k|-|k'|\right)^2\right\}; \quad (27)$$

the integral in Eq. (26) is convergent because the function $Q^0(k,k')$ verifies the properties $Q^0(k,0) = Q^0(0,k') = 0$.

Quite remarkably, one may calculate the integral in Eq. (26) explicitly. To do so, the first step consists in reducing the integration domain $\{(k,k'),\ k+k'\geqslant 0\}$. This domain consists of two sub-domains, $\{(k,k'), k+k'\geqslant 0,\ k\geqslant k'\}$ and $\{(k,k'), k+k'\geqslant 0,\ k'\geqslant k\}$ which yield equal contributions since the integrand is invariant upon interchange of k and k'; hence the integral is equal to twice the integral over the first sub-domain. Also, this sub-domain consists of two sub-sub-domains, $\{(k,k'),\ k\geqslant 0,\ 0\leqslant k'\leqslant k\}$ and $\{(k,k'),\ k\geqslant 0,\ -k\leqslant k'\leqslant 0\}$, over which the function $Q^0(k,k')$ takes the values $kk'/2$ and $-k'(k+k')/2$ respectively; using the change of variable $k''\equiv -k'$ in the integral over the second sub-sub-domain, then re-using the notation k' instead of k'', one finally gets

$$\widetilde{\phi^2}(z) = \frac{2}{\pi^2}\int_0^{+\infty}\left\{\int_0^k\left(\frac{\sin(kd)}{k}\frac{\sin(k'd)}{k'}\frac{\cos((k+k')z)-1}{k+k'}\right.\right.$$
$$\left.\left.+\frac{\sin(kd)}{k^2}\frac{\sin(k'd)}{k'}\left[\cos((k-k')z)-1\right]\right)dk'\right\}dk.$$

In a second step, one may write $k'\equiv\lambda k,\ 0\leqslant\lambda\leqslant 1$ and use the variables of integration (k,λ) instead of (k,k'); the preceding equation then becomes

$$\widetilde{\phi^2}(z) = \frac{2}{\pi^2}\int_0^{+\infty}\left\{\int_0^1\left(\frac{\sin(kd)}{k}\frac{\sin(\lambda kd)}{\lambda k}\frac{\cos((1+\lambda)kz)-1}{(1+\lambda)k}\right.\right.$$
$$\left.\left.+\frac{\sin(kd)}{k^2}\frac{\sin(\lambda kd)}{\lambda k}\left[\cos((1-\lambda)kz)-1\right]\right)kd\lambda\right\}dk$$

or equivalently, upon change of the order of integration,

$$\widetilde{\phi^2}(z) = \frac{2}{\pi^2}\int_0^1\left\{\frac{J[d,\lambda d,(1+\lambda)z]-J(d,\lambda d,0)}{\lambda(1+\lambda)}\right.$$
$$\left.+\frac{J[d,\lambda d,(1-\lambda)z]-J(d,\lambda d,0)}{\lambda}\right\}d\lambda \quad (28)$$

where

$$J(\alpha,\beta,\gamma) \equiv \int_0^{+\infty}\sin(\alpha x)\sin(\beta x)\cos(\gamma x)\frac{dx}{x^2}. \quad (29)$$

The third step consists in calculating the integral $J(\alpha,\beta,\gamma)$; this is done in Appendix A and the result is

$$J(\alpha,\beta,\gamma) = \frac{\pi}{8}(|\alpha+\beta+\gamma|+|\alpha+\beta-\gamma|-|\alpha-\beta+\gamma|-|\alpha-\beta-\gamma|). \quad (30)$$

Eq. (28) becomes, upon use of this formula, replacement of z by $|z|$ (which is admissible since the function $\widetilde{\phi^2}(z)$ is obviously even) and rearrangement of terms,

$$\widetilde{\phi^2}(z) = \frac{d}{4\pi}\int_0^1\left(||1-|u||-2\frac{2+\lambda}{1+\lambda}-\frac{1}{\lambda}\left|\frac{1-\lambda}{1+\lambda}-|u|\right|\right.$$
$$\left.+\frac{1}{\lambda}|1+\lambda-(1-\lambda)|u||\right)d\lambda \quad (31)$$

where u has been defined in Eq. $(24)_2$. In a fourth and final step, one must evaluate this integral; the calculation is straightforward but somewhat heavy because of the presence of absolute values in the integrand which make it necessary to distinguish between cases. The final result reads

$$\widetilde{\phi^2}(z) = \begin{cases} -\frac{d}{2\pi}[(1+u)\ln(1+u)+(1-u)\ln(1-u)] & \text{if } |u|\leqslant 1 \\ -\frac{d}{2\pi}\left[(|u|-1)\ln\left(\frac{|u|+1}{|u|-1}\right)+2\ln 2\right] & \text{if } |u|\geqslant 1. \end{cases} \quad (32)$$

It is remarkable that by Eqs. (24) and $(32)_1$, $\widetilde{\phi^2}(z) = -\widetilde{\phi^1}(z)/2$ inside the obstacle ($|u|\leqslant 1$). Outside of the obstacle ($|u|\geqslant 1$), however, the functions $\widetilde{\phi^1}(z)$ and $\widetilde{\phi^2}(z)$ behave differently; for instance in the limit $|u|\to+\infty$, $\widetilde{\phi^1}(z)$ diverges like $\frac{2d}{\pi}\ln(|u|)$ whereas $\widetilde{\phi^2}(z)$ goes to the constant $-\frac{d}{\pi}(1+\ln 2)$.

Fig. 3 shows the "normalized perturbation of the front" $[\epsilon\phi^1(z)+\epsilon^2\phi^2(z)-\epsilon\phi^1(0)-\epsilon^2\phi^2(0)]/d = [\epsilon\widetilde{\phi^1}(z)+\epsilon^2\widetilde{\phi^2}(z)]/d$, deduced from Eqs. (24) and (32), as a function of the "normalized toughness contrast" ϵ. The nonlinear dependence of $[\epsilon\widetilde{\phi^1}(z)+\epsilon^2\widetilde{\phi^2}(z)]/d$ upon ϵ is quite conspicuous here: the effect of ϵ seems to "saturate" when this parameter becomes large, that is, the linear theory tends to overestimate the deformation of the crack front for large toughness contrasts. This observation seems to be confirmed by experiments performed by Chopin (2010) for a very large toughness contrast of 50, which have produced relatively modest crack front deformations, thus supporting the idea that the crack front becomes stiffer as its perturbation increases in amplitude for a fixed wavelength.

It may be inferred from these numerical results that the range of values of the normalized toughness contrast ϵ for which the above formulae provide reasonably accurate predictions extends up to about 0.2 if Eq. (24) is used alone, and to about 1 if Eqs. (24) and (32) are used in conjunction.

4.2. Case of a periodic distribution of obstacles

The distribution of toughness for a periodic array of infinitely elongated obstacles is represented in Fig. 4, where the period is denoted $2L$. This distribution may be represented by formula (12) with

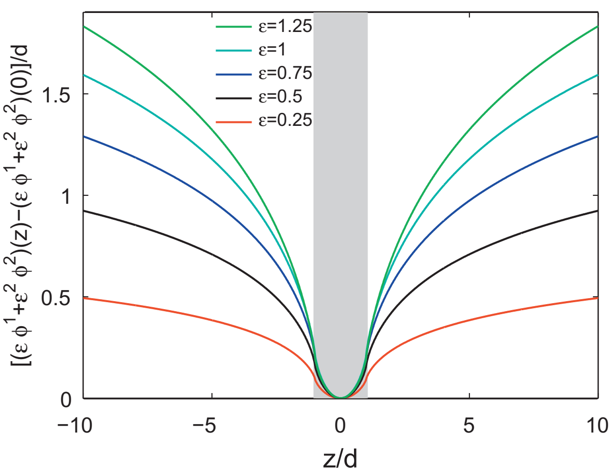

Fig. 3. Equilibrium shape of a crack front penetrating into a single obstacle (represented by a gray rectangle), for various toughness contrasts.

$$\begin{cases} \overline{G_c} \equiv \left(1-\frac{d}{L}\right)G_c^{\mathrm{M}} + \frac{d}{L}G_c^{\mathrm{O}}; \quad \epsilon \equiv \frac{G_c^{\mathrm{O}}-G_c^{\mathrm{M}}}{\overline{G_c}}; \\ g_c(x,z) \equiv g_c(z) \equiv \begin{cases} 1-\frac{d}{L} & \text{if } |z-2nL|<d \\ -\frac{d}{L} & \text{if } |z-2nL|>d \end{cases} ((2n-1)L \leqslant z \leqslant (2n+1)L). \end{cases} \tag{33}$$

The functions g_c and $\hat{g_c}$ are then of the equivalent forms

$$g_c(x,z) \equiv \sum_{m=-\infty}^{+\infty} c_m e^{imk_0 z} \Longleftrightarrow \hat{g_c}(x,k) \equiv \sum_{m=-\infty}^{+\infty} c_m \delta(k-mk_0) \tag{34}$$

where

$$k_0 \equiv \frac{\pi}{L}; \quad c_m \equiv \frac{1}{2L}\int_{-L}^{L} g_c(z)e^{-im\pi z/L}dz = \begin{cases} 0 & \text{if} \quad m=0 \\ \frac{\sin(m\pi d/L)}{m\pi} & \text{if} \quad m \neq 0. \end{cases} \tag{35}$$

At order 1, Eqs. (17), $(34)_2$ and (35) yield

$$\begin{aligned} \hat{\phi}^1(a;k) &= -\sum_{m=-\infty}^{+\infty} \frac{c_m}{\frac{|m|\pi}{L} - \frac{dG^0}{G^0da}(a)} \delta\left(k-\frac{m\pi}{L}\right) \\ \Rightarrow \phi^1(a;z) & \\ &= -\sum_{m=-\infty}^{+\infty} \frac{c_m}{\frac{|m|\pi}{L} - \frac{dG^0}{G^0da}(a)} e^{im\pi z/L} \\ &= -\sum_{m\neq 0} \frac{\sin(m\pi d/L)}{m\pi\left[\frac{|m|\pi}{L} - \frac{dG^0}{G^0da}(a)\right]} e^{im\pi z/L} \\ &= -2\sum_{m=1}^{+\infty} \frac{\sin(m\pi d/L)}{m\pi\left[\frac{m\pi}{L} - \frac{dG^0}{G^0da}(a)\right]} \cos(m\pi z/L). \end{aligned}$$

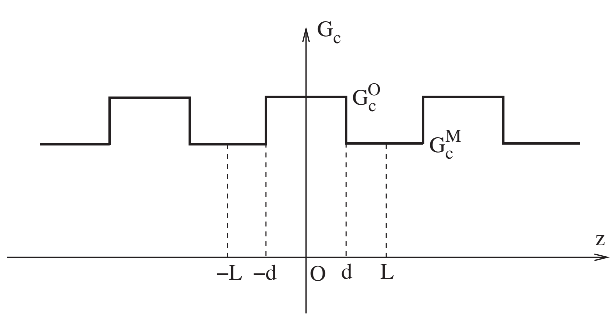

Fig. 4. The distribution of fracture toughness in an infinite body containing a periodic array of obstacles.

Taking the limit $\frac{dG^0}{da}(a) \to 0$ does not raise any particular problem here; $\phi^1(a;z)$ goes to a limit $\phi^1(z)$ given by the following series, which unfortunately cannot be expressed in terms of elementary functions:

$$\phi^1(z) \equiv -\frac{2L}{\pi^2}\sum_{m=1}^{+\infty} \frac{\sin(m\pi d/L)}{m^2}\cos(m\pi z/L). \tag{36}$$

At order 2, one gets from Eqs. (18), $(34)_2$ and (35):

$$\begin{aligned} \phi^2(a;z) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q(a;k,k') \frac{\hat{g_c}(k)}{|k| - \frac{dG^0}{G^0da}(a)} \frac{\hat{g_c}(k')}{|k'| - \frac{dG^0}{G^0da}(a)} \\ &\quad \times \frac{e^{i(k+k')z}}{|k+k'| - \frac{dG^0}{G^0da}(a)} dkdk' \\ &= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} Q\left(a;\frac{m\pi}{L},\frac{n\pi}{L}\right) \frac{c_m}{\frac{|m|\pi}{L} - \frac{dG^0}{G^0da}(a)} \frac{c_n}{\frac{|n|\pi}{L} - \frac{dG^0}{G^0da}(a)} \\ &\quad \times \frac{e^{i(m+n)\pi z/L}}{\frac{|m+n|\pi}{L} - \frac{dG^0}{G^0da}(a)} \\ &= \sum_{m\neq 0, n\neq 0} Q\left(a;\frac{m\pi}{L},\frac{n\pi}{L}\right) \frac{\sin(m\pi d/L)}{m\pi\left[\frac{|m|\pi}{L} - \frac{dG^0}{G^0da}(a)\right]} \\ &\quad \times \frac{\sin(n\pi d/L)}{n\pi\left[\frac{|n|\pi}{L} - \frac{dG^0}{G^0da}(a)\right]} \frac{e^{i(m+n)\pi z/L}}{\frac{|m+n|\pi}{L} - \frac{dG^0}{G^0da}(a)}. \end{aligned}$$

Subtracting the average value $\langle\phi^2\rangle$ of ϕ^2, corresponding to the terms having $m+n=0$, one gets from there

$$\begin{aligned} \overline{\phi^2}(a;z) &\equiv \phi^2(a;z) - \langle\phi^2\rangle \\ &= \sum_{m\neq 0, n\neq 0, m+n\neq 0} Q\left(a;\frac{m\pi}{L},\frac{n\pi}{L}\right) \frac{\sin(m\pi d/L)}{m\pi\left[\frac{|m|\pi}{L} - \frac{dG^0}{G^0da}(a)\right]} \\ &\quad \times \frac{\sin(n\pi d/L)}{n\pi\left[\frac{|n|\pi}{L} - \frac{dG^0}{G^0da}(a)\right]} \frac{e^{i(m+n)\pi z/L}}{\frac{|m+n|\pi}{L} - \frac{dG^0}{G^0da}(a)} \\ &= \sum_{m\neq 0, n\neq 0, m+n>0} 2Q\left(a;\frac{m\pi}{L},\frac{n\pi}{L}\right) \frac{\sin(m\pi d/L)}{m\pi\left[\frac{|m|\pi}{L} - \frac{dG^0}{G^0da}(a)\right]} \\ &\quad \times \frac{\sin(n\pi d/L)}{n\pi\left[\frac{|n|\pi}{L} - \frac{dG^0}{G^0da}(a)\right]} \frac{\cos[(m+n)\pi z/L]}{\frac{(m+n)\pi}{L} - \frac{dG^0}{G^0da}(a)}. \end{aligned}$$

In the limit $\frac{dG^0}{da}(a) \to 0$, this expression goes to a limit $\overline{\phi^2}(z)$ given by

$$\begin{aligned} \overline{\phi^2}(z) &\equiv \frac{2L^3}{\pi^5}\sum_{m\neq 0, n\neq 0, m+n>0} Q^0\left(\frac{m\pi}{L},\frac{n\pi}{L}\right) \frac{\sin(m\pi d/L)}{m|m|} \times \frac{\sin(n\pi d/L)}{n|n|} \\ &\quad \times \frac{\cos[(m+n)\pi z/L]}{m+n}. \end{aligned}$$

Accounting finally for the expression (27) of the function $Q^0(k,k')$, one gets from there, after a few transformations:

$$\begin{aligned} \overline{\phi^2}(z) &= \frac{L}{\pi^3}\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} \frac{\sin(m\pi d/L)}{m(m+n)}\left[\frac{\sin(n\pi d/L)}{n}\cos[(m+n)\pi z/L]\right. \\ &\quad \left. +2\frac{\sin[(m+n)\pi d/L]}{m+n}\cos(n\pi z/L)\right]. \end{aligned} \tag{37}$$

Eqs. (36) and (37) permit to calculate the normalized perturbation of the front $[\epsilon\phi^1(z) + \epsilon^2\overline{\phi^2}(z) - \epsilon\phi^1(0) - \epsilon^2\overline{\phi^2}(0)]/d = [\epsilon\phi^1(z) + \epsilon^2\overline{\phi^2}(z) - \epsilon\phi^1(0) - \epsilon^2\overline{\phi^2}(0)]/d$ numerically, as a function of the normalized toughness contrast ϵ and the dimensionless parameter L/d comparing the respective sizes of the period and the obstacle. Figs. 5 and 6 show the results obtained; Fig. 5 is for $\epsilon = 1$ and various values of L/d, and Fig. 6 for $L/d = 3$ and various values of ϵ. One sees in Fig. 5 that the influence of the finiteness of the period is maximum half-way between the obstacles, where the slope of the

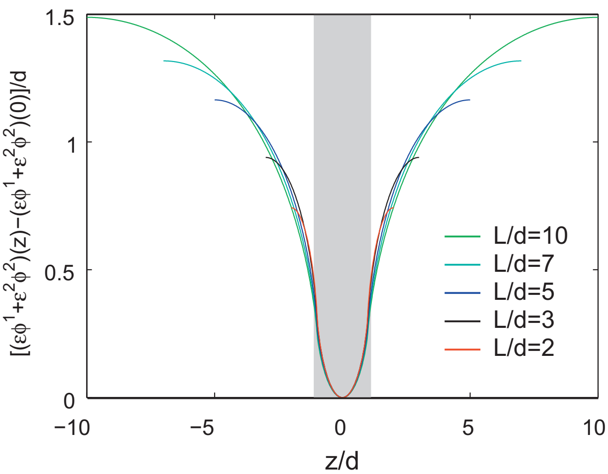

Fig. 5. Equilibrium shape of a crack front penetrating into a periodic array of obstacles – Influence of the relative sizes of the period and the obstacle for a fixed toughness contrast ($\epsilon = 1$).

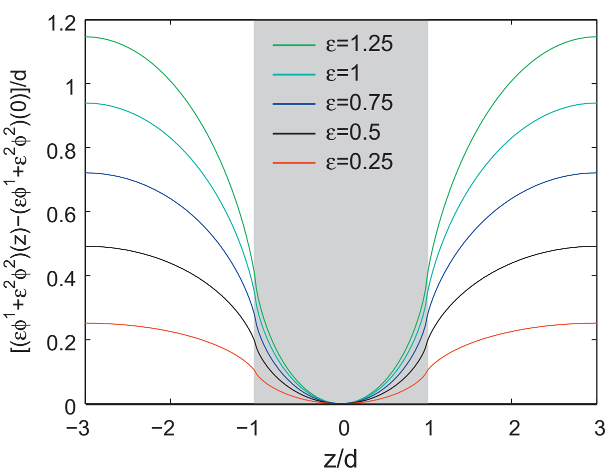

Fig. 6. Equilibrium shape of a crack front penetrating into a periodic array of obstacles – Influence of the toughness contrast for fixed relative sizes of the matrix and the obstacle ($L/d = 3$).

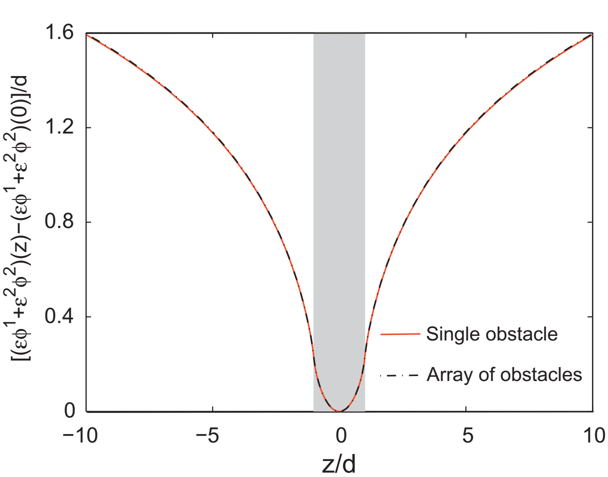

Fig. 7. Comparison of equilibrium shapes of crack fronts penetrating into a single obstacle and a periodic array of obstacles with a large period ($L/d = 640$), for a normalized toughness contrast $\epsilon = 1$.

front is zero for symmetry reasons. One also sees in Fig. 6 that in the periodic case, the nonlinear effect of the toughness contrast is weak, except near the boundary of the obstacles (observe that at $z/d = \pm 1$, the gap between the red and green curves, corresponding to values of ϵ differing by a factor of 5, is lower than what the linear theory would predict).

Finally, it is worth noting that in the limit of a very large period, the results for a periodic array of obstacles converge toward those for a single one; this is illustrated in Fig. 7 which compares crack front shapes obtained for a single obstacle and a periodic array having $L/d = 640$, for a normalized toughness contrast ϵ of unity.

The theoretical predictions of Eqs. (24) and (32) for an isolated pinning obstacle on the one hand, and (36) and (37) for a periodic array of obstacles on the other hand, provide a basis to interpret crack front patterns observed in heterogeneous interfaces (see e.g. Dalmas et al., 2009; Chopin et al., 2011; Xia et al., 2012). In particular, it provides an efficient means of measuring the toughness of defects or impurities in these systems, using the toughness contrast as an adjustable parameter in the fit of experimental data.

5. Conclusion

The aim of this work was to apply Leblond et al. (2012)'s expression of the stress intensity factor up to second order in the perturbation of the crack front to investigate geometrically nonlinear effects in the quasistatic, coplanar propagation of cracks in heterogenous materials.

In a first step, this required an extension of Leblond et al. (2012)'s treatment to the case where the unperturbed SIF $K^0(a)$, for the straight configuration of the front, depends on its position a within the crack plane. Indeed when $dK^0/da = 0$, mathematical difficulties arise in the form of divergent integrals appearing in the expression of the equilibrium shape of the crack front resulting from a given distribution of fracture toughness. Consideration of this interesting limiting case therefore requires to first perform all calculations with $dK^0/da \neq 0$, before finally (and carefully) taking the limit $dK^0/da \to 0$. In addition, the new formula extends the second order expression of the stress intensity factor of a perturbed crack front to more realistic fracture tests geometries.

In a second step, the formulae obtained for the first- and second-order variations of the SIF resulting from a given perturbation of the front were applied to the explicit calculation of the equilibrium shape of the front of a crack propagating in a heterogeneous material according to Griffith's criterion, up to second order in the fluctuations of fracture toughness.

As an application, we calculated the equilibrium shape of a crack front penetrating into a single obstacle of infinite length in the direction of propagation, or a periodic array of such obstacles, up to second order in the contrast of toughness between the matrix and the obstacle (s). The second-order perturbation of the front was expressed in a very simple analytical form for a single obstacle, and as an infinite double series for an array of obstacles. The formulae obtained may be useful in the future to study the pinning of cracks by heterogeneities in experimental situations.

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Appendix A. Calculation of the integral $\boldsymbol{J}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$

The well-known trigonometric relations

$$\begin{cases} \cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)] \\ \sin a \sin b = \frac{1}{2}[-\cos(a+b) + \cos(a-b)] \end{cases}$$

imply that

$$\sin(\alpha x)\sin(\beta x)\cos(\gamma x) = \frac{1}{2}\{-\cos[(\alpha+\beta)x] + \cos[(\alpha-\beta)x]\}\cos(\gamma x)$$
$$= \frac{1}{4}\{-\cos[(\alpha+\beta+\gamma)x] - \cos[(\alpha+\beta-\gamma)x] + \cos[(\alpha-\beta+\gamma)x] + \cos[(\alpha-\beta-\gamma)x]\}.$$

It then follows from Eq. (29) that

$$J(\alpha,\beta,\gamma) = \frac{1}{4}\int_0^{+\infty}\{-\cos[(\alpha+\beta+\gamma)x] - \cos[(\alpha+\beta-\gamma)x] + \cos[(\alpha-\beta+\gamma)x] + \cos[(\alpha-\beta-\gamma)x]\}\frac{dx}{x^2} = \frac{1}{4}[I(\alpha+\beta+\gamma) + I(\alpha+\beta-\gamma) - I(\alpha-\beta+\gamma) - I(\alpha-\beta-\gamma)],$$

$$I(\lambda) \equiv \int_0^{+\infty}\frac{1-\cos(\lambda x)}{x^2}dx.$$

The integral $I(\lambda)$ is given by formula (3.782.2) of Gradshteyn and Ryzhik (1980):

$$I(\lambda) = \frac{\pi}{2}|\lambda|.$$

Inserting this result into the preceding expression of $J(\alpha,\beta,\gamma)$, one gets Eq. (30) of the text.

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